Final exam for CSI35

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1. Let n be a positive integer. Prove using mathematical induction:

$$1^3 + 2^3 + \cdots n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

2. Provide a simple formula that generates the terms of the sequence that begins with

$$-2, -1, 2, 7, 14, 23.$$

3. If w is a string then its *reverse* w^R is the string obtained by reading w backwards, for example the reverse of the string *sub* is the string *bus*.

   (a) Give an inductive definition of the reverse of a string.

   (b) Use structural induction to prove that for all strings w_1 and w_2 the following holds:

   $$(w_1 w_2)^R = w_2^R w_1^R$$

4. Let $\mathbb{M}_3$ be the set of $n \times n$ matrices and Δ_3 be the set of *upper triangular* 3×3 matrices:

   $$\Delta_3 = \{(a_{ij}) \in \mathbb{M}_3 \,|\, a_{21} = a_{31} = a_{32} = 0\}$$

   Consider the following relation on $\mathbb{M}_n$

   $$R = \{(A, B) \in \mathbb{M}_3^2 \,|\, A - B \in \Delta_3\}$$

   (a) Prove that R is an equivalence relation.

   (b) Find the equivalence class of

   $$A = \begin{pmatrix} 1 & -4 & 5 \\ -3 & 9 & 7 \\ 5 & 7 & 3 \end{pmatrix}$$

5. Draw the Hasse diagram for the relation of divisibility on the set $\{1, 2, \ldots, 20\}$.

6. Consider the poset $(\mathbb{Z}^+, |)$.

   (a) Find all upper bounds of the set $\{4, 6\}$. Is there a least upper bound?

   (b) Find all lower bounds of the set $\{8, 12\}$. Is there a greatest lower bound?

7. Refer to Figure 1. Are the graphs (a) and (b) isomorphic? How about (c) and (d)? Justify your answer, in particular if your answer is affirmative you should provide an isomorphism.
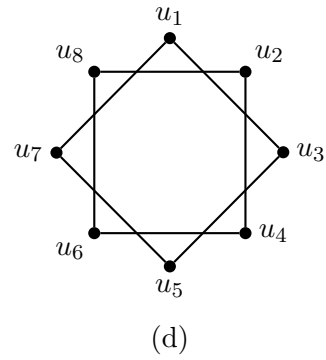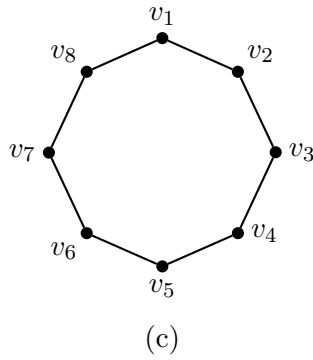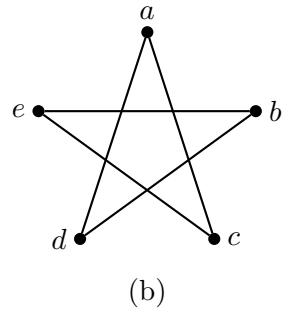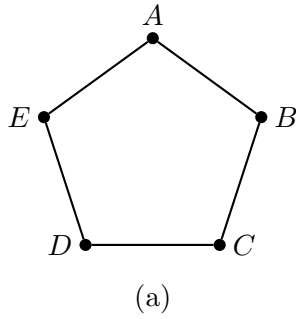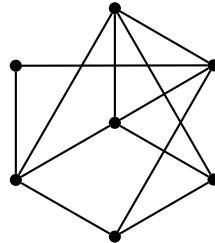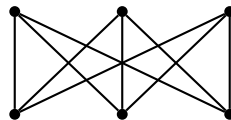
Figure 1:

8. Does the following graph have an Euler circuit? Does it have an Euler path?

9. Does the following graph have an Hamilton circuit? Does it have an Hamilton path?

10. List all possible trees with five or less vertices.

11. Draw the game tree for the game of nim if the starting position consists of three piles with one, two and three stones respectively. Which player has a winning strategy?

12. How many children does the root of the game tree for nim have if the starting position consists of three piles with seven, five and three stones respectively?

13. Give a solution to the Eight-Queens puzzle.