Answers to the second exam for Math 13

Nikos Apostolakis

1. Find the domain of the function $f(x) = \ln(\ln(3x))$.

   Answer. The domain of a logarithmic function is the set of all positive real numbers $\{x > 0\}$, and therefore we need the following two conditions to hold:

   1. In order for $\ln 3x$ to make sense we need to have $3x > 0$, and
   2. in order for $\ln(\ln(3x))$ to make sense we need $\ln(3x) > 0$.

      From the first condition we obtain:

      $$x > 0 \tag{1}$$

      For the second condition we note that $\ln x > 0 \iff x > 1$ and therefore in order for the second condition to be satisfied we need $3x > 1$ or equivalently

      $$x > \frac{1}{3} \tag{2}$$

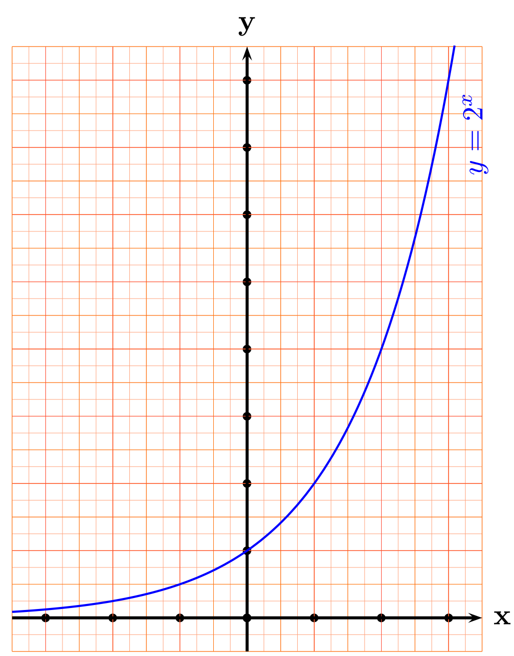
      Now if (2) holds then (1) holds as well. We conclude that the domain of $f(x)$ is

      $$\{x > \frac{1}{3}\}.$$

   □

2. Sketch a graph of the function $y = 2^x$.

   Answer.



3. Express as a sum difference or multiple of logarithms: $\log_2\left(\frac{(2x-4)^6\sqrt[5]{x^2-4y}}{x^3y^2}\right)$

Answer. We have:

$$\log_2\left(\frac{(2x-4)^6\sqrt[5]{x^2-4y}}{x^3y^2}\right) = \log_2\left((2x-4)^6\sqrt[5]{x^2-4y}\right) - \log_2 x^3y^2$$
$$= \log_2(2x-4)^6 + \log_2\left(\sqrt[5]{x^2-4y}\right) - \left(\log_2 x^3 + \log_2 y^2\right)$$
$$= 6\log_2(2x-4) + \log_2\left(x^2-4y\right)^{1/5} - \log_2 x^3 - \log_2 y^2$$
$$= 6\log_2(2x-4) + \frac{1}{5}\log_2(x^2-4y) - 3\log_2 x - 2\log_2 y$$

□

4. Solve for y: $\log(y^5-2) - \log(x^2-5) + 3\log(7x+6) = 1\,.$

Answer. We have:

$$\log(y^5-2) - \log(x^2-5) + 3\log(7x+6) = 1 \iff \log(y^5-2) - \log(x^2-5) + \log(7x+6)^3 = \log 10$$
$$\iff \log\frac{(y^5-2)(7x+6)^3}{x^2-5} = \log 10$$
$$\iff \frac{(y^5-2)(7x+6)^3}{x^2-5} = 10$$
$$\iff y^5 - 27 = \frac{10(x^2-5)}{(x+6)^3}$$
$$\iff y^5 = \frac{10(x^2-5)}{(x+6)^3} + 27$$
$$\iff y = \sqrt[5]{\frac{10(x^2-5)}{(x+6)^3} + 27}$$

□

5. Solve: $e^{x^5 - \log_5 \tan x + 3\sin^2 x} = \log_2\left(\frac{1}{2}\right)$.

Answer. Since $\log_2\left(\frac{1}{2}\right) = -1$ the given equation is equivalent to

$$e^{x^5 - \log_5 \tan x + 3\sin^2 x} = -1$$

And this equation has no solutions since for all real numbers x we have that $e^x > 0$. □

6. Solve: $4^x = 5^{x-2}$.

Answer. Taking natural logarithm of both sides we get:

$$\begin{aligned} 4^x = 5^{x-2} &\iff \ln 4^x = \ln 5^{x-2} \\ &\iff x \ln 4 = (x-2) \ln 5 \\ &\iff x \ln 4 = x \ln 5 - 2 \ln 5 \\ &\iff 2 \ln 5 = x \ln 5 - x \ln 4 \\ &\iff 2 \ln 5 = x(\ln 5 - \ln 4) \\ &\iff \frac{2 \ln 5}{\ln 5 - \ln 4} = x \end{aligned}$$

Now substituting the values of the logarithms we obtain

$$x \approx \frac{3.21887582486}{1.609 - 1.386} \approx 14.425$$

□

7. Solve: $2^x - 12 \cdot (2^{-x}) = 1$

Answer. Let $y = 2^x$ then $2^{-x} = y^{-1}$ and the given equation transforms to

$$y + \frac{12}{y} = 1$$

or equivalently after multiplying with y,

$$y^2 + 12 = y$$

and finally,

$$y^2 - y + 12 = 0$$

The last equation is equivalent to

$$(y-4)(y+3) = 0$$

and this gives

$$y = 4 \text{ or } y = -3$$

So we have that

$$2^x = 4 \text{ or } 2^x = -3\,.$$

The second equation is impossible because 2^x is always positive so we conclude that

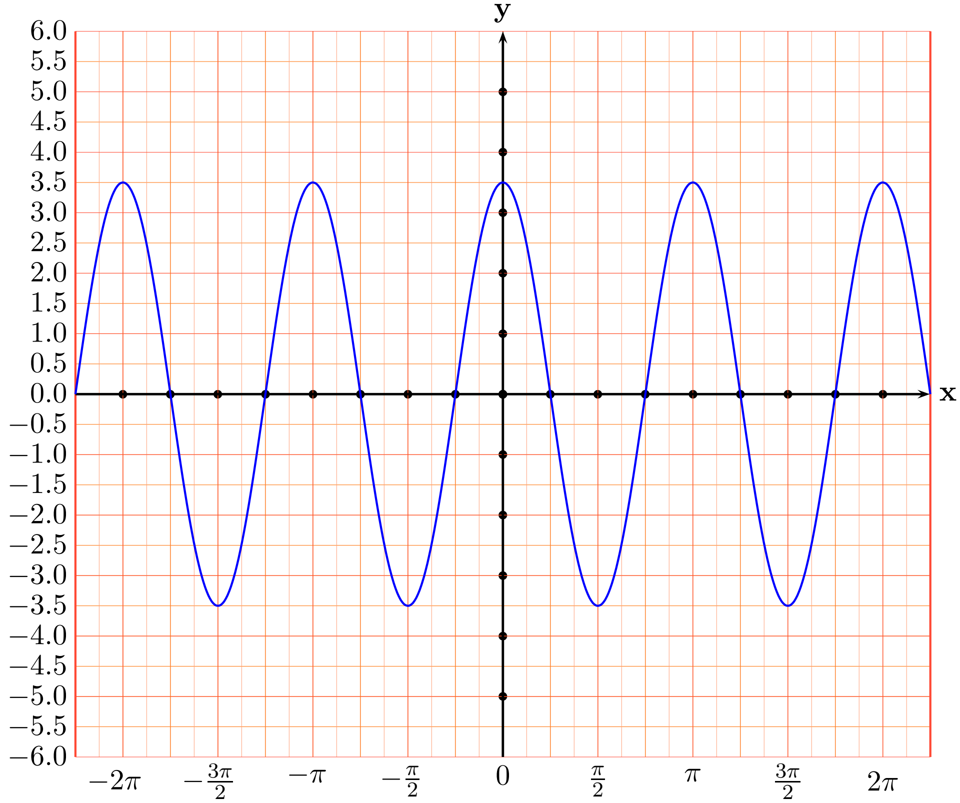
$$2^x = 4$$

or equivalently

$$x = 2\,.$$

□

8. Find an equation of the following *sinusoidal* curve:



Answer. The amplitude of the curve is 3.5 and in an interval of length 2π the curve completes two full cycles. Thus the equation of the curve has the form

$$y = 3.5\sin(2x + d) \tag{3}$$

where the displacement d is to be determined. The curve with equation (3) meets the x–axis when

$$2x + d = 0$$

or equivalently when

$$x = -\frac{d}{2}$$

Now the given curve meats the x–axis when $x = -\pi/4$ and therefore we may assume that

$$-\frac{d}{2} = -\frac{\pi}{4}$$

and thus

$$d = \frac{\pi}{2}$$

So the equation of the curve is

$$y = 3.5\sin(2x + \frac{\pi}{2})$$

Remark. Notice that this equation is equivalent to

$$y = 3.5\cos 2x$$

9. A particle moves on a circle of radius 10 starting at at time $t = 0$ at the initial angle of $\pi/3\,\text{rad}$ and with constant angular velocity $\omega = 3.00\,\text{rad/s}$. Write an equation that describes the displacement of the projection of the particle on the y-axis.

Answer. The equation of this motion is given by

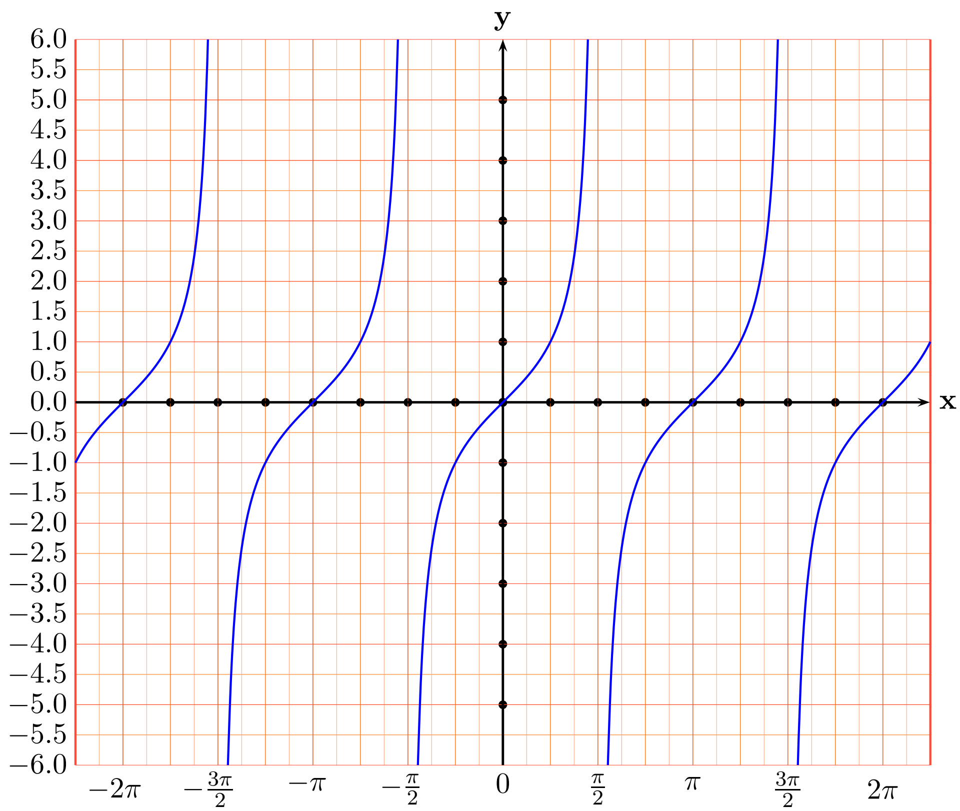
$$y = R\sin(\omega t + \theta)$$

Substituting the given values we get

$$y = 10\sin(3t + \frac{\pi}{3})$$

□

10. Sketch a graph of $y = \tan x$. *Answer.*



11. Prove: $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$.

Answer. We have that

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Multiplying these two equations we get

$$\cos(x+y)\cos(x-y) = \cos^2 x \cos^2 y - \sin^2 x \sin^2 y$$

Now use the identity $\sin^2 x = 1 - \cos^2 x$ to get

$$\begin{aligned}
\cos^2 x \cos^2 y - \sin^2 x \sin^2 y &= \cos^2 x \cos^2 y - (1 - \cos^2 x)\sin^2 y \\
&= \cos^2 x \cos^2 y - \sin^2 y + \cos^2 x \sin^2 y \\
&= \cos^2 x(\cos^2 y + \sin^2 y) - \sin^2 y \\
&= \cos^2 x - \sin^2 y
\end{aligned}$$

□